P144

15. Sol:

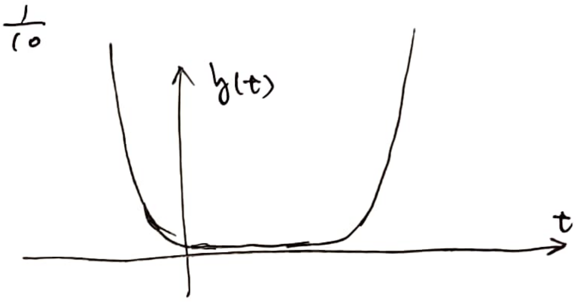
the characteristic equation is:

$$\lambda^2 + 8\lambda - 9 = 0 \quad \Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -9$$

i.e. $y(t) = C_1 e^{t-1} + C_2 e^{-9(t-1)}$

$$\left.\begin{aligned} y(1) &= C_1 + C_2 = 1 \\ y'(1) &= C_1 - 9C_2 = 0 \end{aligned}\right\} \Rightarrow \begin{cases} C_1 = \frac{9}{10} \\ C_2 = \frac{1}{10} \end{cases}$$

i.e. $y(t) = \frac{9}{10} e^{t-1} + \frac{1}{10} e^{-9(t-1)}$




$y \to \infty$ as $t \to \infty$.

21. Sol:

Chara equ: $\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = 2.$

$y(t) = C_1 e^{-t} + C_2 e^{2t}$

$$\begin{cases} y(0) = C_1 + C_2 = \alpha \\ y'(0) = -C_1 + 2C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{2}{3}(\alpha - 1) \\ C_2 = \frac{\alpha + 2}{3} \end{cases}$$

<u>want</u>: $y(\infty) = 0$, i.e. $C_2 = 0 \Rightarrow \alpha = -2$.

P155.

6. Sol:

$$W = \begin{vmatrix} \cos^2\theta & 1 + \cos 2\theta \\ -2\cos\theta\sin\theta & -2\sin 2\theta \end{vmatrix} = \begin{vmatrix} \frac{1+\cos 2\theta}{2} & 1 + \cos 2\theta \\ -\sin 2\theta & -2\sin 2\theta \end{vmatrix} = 0$$

15. Sol:

$$[c\varphi(t)]'' + p(t)[c\varphi(t)]' + q(t)[c\varphi(t)]$$

$$= c\left[\varphi''(t) + p(t)\varphi'(t) + q(t)\varphi(t)\right] = c\,g(t).$$

Since $g(t) \not\equiv 0$, then $c\varphi(t)$ is not a sol.

which is because the equation is inhomogeneous.

16. Sol:

$$y'(t) = \frac{d}{dt}\sin t^2 = 2t\cos t^2$$

$$y''(t) = \frac{d}{dt}(2t\cos t^2) = 2\cos t^2 - 4t^2\sin t^2.$$

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t)$$

$$\cancel{2t\cos t^2 + p(t)}$$

$$= 2\cos t^2 - 4t^2\sin t^2 + p(t)\,2t\cos t^2 + q(t)\sin t^2 = 0.$$

$\Rightarrow p(t) = \frac{1}{t}$, $q(t) = 4t^2$, which needs $t \neq 0$.

Hence, the answer is "NO".

18. Sol:

$$W(f,g) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = t g'(t) - g(t) = t^2 e^t.$$

$$\Rightarrow \frac{g'(t)}{t} - \frac{g(t)}{t^2} = e^t$$

$$\Rightarrow \frac{d}{dt}\left[\frac{g(t)}{t}\right] = e^t \quad \Rightarrow \quad \frac{g(t)}{t} = e^t + c$$

i.e. $g(t) = te^t + ct$.

21. Sol:

$$W(y_3, y_4) = \begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix} = \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix}$$

$$= \begin{vmatrix} a_1y_1 & b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} + \begin{vmatrix} a_2y_2 & b_1y_1 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix}$$

$$= \begin{vmatrix} a_1y_1 & b_2y_2 \\ a_1y_1' & b_2y_2' \end{vmatrix} + \begin{vmatrix} a_1y_1 & b_2y_2 \\ a_2y_2' & b_1y_1' \end{vmatrix} + \begin{vmatrix} a_2y_2 & b_1y_1 \\ a_2y_2' & b_1y_1' \end{vmatrix} + \begin{vmatrix} a_2y_2 & b_1y_1 \\ a_1y_1' & b_2y_2' \end{vmatrix}$$

$$= a_1b_2 \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} + a_1b_1y_1y_1' - a_2b_2y_2y_2' + a_2b_1 \begin{vmatrix} y_2 & y_1 \\ y_2' & y_1' \end{vmatrix} + a_2b_2y_2y_2' - a_1b_1y_1y_1'$$

$$= (a_1b_2 - a_2b_1) \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (a_1b_2 - a_2b_1)\, W(y_1, y_2).$$

26. Sol:

$y_1' = 1, \quad y_1'' = 0$

then $x^2 y_1'' - x(x+2)y_1' + (x+2)y_1 = 0$ ✓

$y_2' = e^x + xe^x \qquad y_2'' = 2e^x + xe^x$

$\Rightarrow x^2 y_2'' - (x+2)x\,y_2' + (x+2)y_2$

$= (2+x)x^2e^x - x(x+2)(x+1)e^x + x(x+2)e^x$

$= (x+2)e^x\left(x^2 - x(x+1) + x\right) = 0.$ ✓

since $y_1$ & $y_2$ are linear indep. they are fundamental set of sol's.

32. Sol:

$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$

$\Rightarrow y'' - \frac{2x}{1-x^2}y' + \alpha(\alpha+1)y = 0.$

$\Rightarrow W(t) = C\exp\left\{\int \frac{2x}{1-x^2}dx\right\} = C\exp\left\{-\log(1-x^2)\right\} = \frac{C}{1-x^2}$

36. Sol:

$$W(t) = C \exp\left\{-\int p(t)\,dt\right\} = \text{const}$$

$$\Rightarrow \int p(t)\,dt = \text{const} \Rightarrow p \equiv 0.$$

37. Sol:

$$W(fg, fh) = \begin{vmatrix} fg & fh \\ (fg)' & (fh)' \end{vmatrix}$$

$$= \begin{vmatrix} fg & fh \\ fg' & fh' \end{vmatrix} + \begin{vmatrix} fg & fh \\ f'g & f'h \end{vmatrix}$$

$$= f^2 W(g, h)$$

P164.

11. Sol:

Char equ: $\lambda^2 + 6\lambda + 13 = 0$

$$\Rightarrow \lambda_1 = -3 + 2i \qquad \lambda_2 = -3 - 2i.$$

i.e. $y_1(x) = e^{-3x}\sin 2x \qquad y_2(x) = e^{-3x}\cos 2x$

$$\Rightarrow y(x) = C_1 e^{-3x}\sin 2x + C_2 e^{-3x}\cos 2x.$$

12. Sol:

Char equ: $4\lambda^2 + 9 = 0$

$$\Rightarrow \lambda_1 = \frac{3}{2}i \qquad \lambda_2 = -\frac{3}{2}i$$

i.e. $y_1(x) = \sin\frac{3}{2}x \qquad y_2(x) = \cos\frac{3}{2}x$

$$\Rightarrow y(x) = C_1 \sin\frac{3}{2}x + C_2 \cos\frac{3}{2}x.$$

33. Sol:

Suppose $t_1$ & $t_2$ are two zeros of $y_1$, between which there are no zeros of $y_2$.

Then $\frac{y_1}{y_2}$ is well defined on $(t_1, t_2)$, differentiable

and $\frac{y_1}{y_2}(t_1) = \frac{y_1}{y_2}(t_2)$.

By Rolle's thm, $\exists\ t_0 \in (t_1, t_2)$ s.t. $\frac{d}{dt}\left(\frac{y_1}{y_2}\right)(t_0) = 0$.

i.e. $\frac{y_1' y_2 - y_2' y_1}{y_2^2}(t_0) = 0 \Rightarrow W(y_1, y_2)(t_0) = 0$.

which contradictory to the fact that $y_1$ & $y_2$ are fundamental set of sol's to $y'' + p(t)y' + q(t)y = 0$.

34. Sol:

1> set $x = \log t$,

then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dx}{dt}$$

$$= \frac{d}{dx}\left[\frac{dy}{dx}\frac{dx}{dt}\right]\frac{dx}{dt} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d}{dx}\left(\frac{dx}{dt}\right)\frac{dx}{dt}$$

$$= \frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx},$$ since $\frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dx}(e^{-x})$

$= -e^{-x} = -\frac{1}{t}$.

b> $t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y$

$$= \frac{d^2y}{dx^2} - \frac{dy}{dx} + \alpha\frac{dy}{dx} + \beta y$$

$$= \frac{d^2y}{dx^2} + (\alpha-1)\frac{dy}{dx} + \beta y = 0.$$

40. Sol:

set $x = \log t$, then the equ becomes.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0.$$

$$\Rightarrow \quad y = c_1 e^x \cos x + c_2 e^x \sin x$$

$$\Rightarrow \quad y(t) = c_1 t \cos(\log t) + c_2 t \sin(\log t).$$

43. Sol:

a) $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dx}{dt}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right)$$

$$= \frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \frac{dx}{dt}\frac{d^2y}{dx^2}\frac{dx}{dt} + \frac{dy}{dx}\frac{d^2x}{dt^2}$$

$$= \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

b) $$y''(t) + p(t)y'(t) + q(t)y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{d^2x}{dt^2}\frac{dy}{dx} + p(t)\frac{dx}{dt}\frac{dy}{dx} + q(t)y = 0$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0.$$

d) notice that $\frac{dx}{dt} = (q(t))^{\frac{1}{2}}$

then $$\frac{\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}}{\left(\frac{dx}{dt}\right)^2} = \frac{\frac{1}{2}q'(t)(q(t))^{-\frac{1}{2}} + p(t)(q(t))^{\frac{1}{2}}}{q(t)}$$

$$= \frac{q'(t) + 2p(t)q(t)}{2(q(t))^{\frac{3}{2}}} = \text{const}.$$

if $q < 0$, take $x := \int \sqrt{-q}\,dt$